University of Oxford

Mathematical Institute

InFoMM
Industrially Focused
Mathematical Modelling

Poor players or dodgy design: How important is the choice of football in the modern game?

Neil Chada, Peter De Ford, Caoimhe Rooney, Bogdan Toader, Florian Wechsung and Tim Whitbread

Mentor: Dr Timothy Reis

Introduction

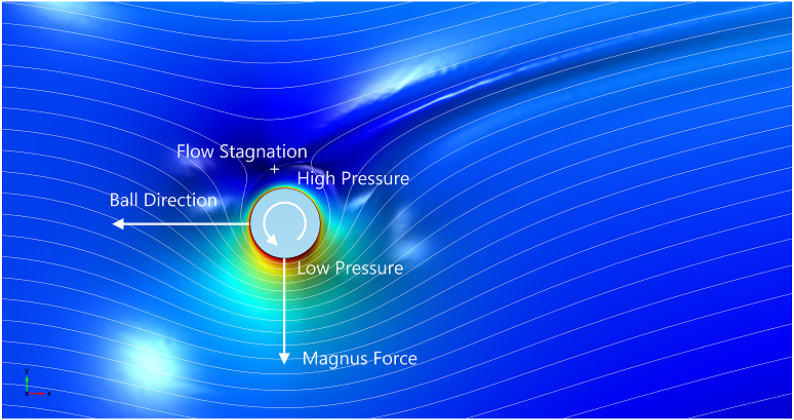
During the Football World Cup in 2010 the new ball introduced by Adidas displayed a weird behaviour. Players were confused about the trajectory of the ball, which leads to the following questions:

- Why does it swerve?
- Can we model the flight of a football?
- Does the choice of ball matter?
- Does the location of the game matter?

Why does a ball swerve?

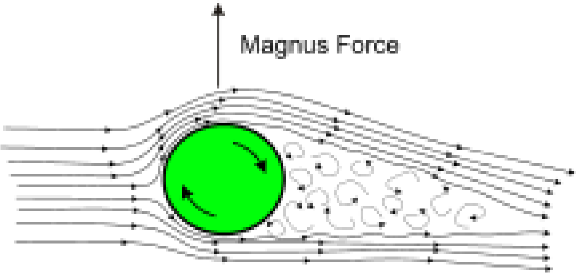
Bernoulli's Theorem for an inviscid flow:

$$\frac{|\mathbf{v}|^2}{2} + p = const.$$



Why does a ball swerve?

The boundary layer on the side of the ball with a greater velocity becomes turbulent and separates from the ball later than the laminar boundary layer. The wake then becomes deflected and so the ball is ‘pushed’ towards the direction of spin.



Equations

- Magnus Force:
$$F_L = \frac{8}{3}\pi r^2 \rho C_L |v|^2,$$
where ρ is air density, v is velocity, r is the radius of the ball and C_L is the dimensionless lift coefficient.
- Drag force:
$$F_D = \frac{1}{2}\pi r^2 \rho C_D |v|^2,$$
where C_D is the dimensionless drag coefficient.
- It is important to note that the Magnus force acts perpendicular to the drag force, which opposes the direction of travel.

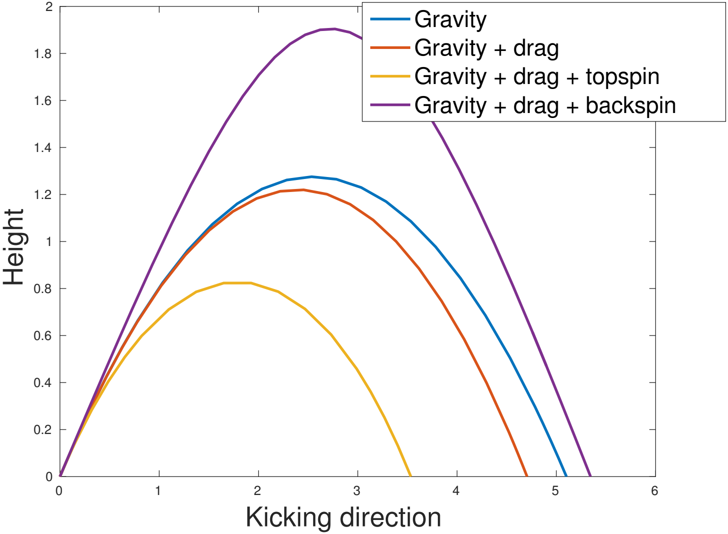
- Considering motion in two-dimensions, we can analyse the path of the ball by looking from above and from the side.
- Looking from the side, we consider the following model:

$$\begin{aligned}\ddot{x} &= \quad -\frac{1}{m}\frac{\dot{x}}{|\dot{\mathbf{x}}|}F_D - \frac{1}{m}\frac{\dot{z}}{|\dot{\mathbf{x}}|}F_L \\ \ddot{z} &= -g - \frac{1}{m}\frac{\dot{z}}{|\dot{\mathbf{x}}|}F_D + \frac{1}{m}\frac{\dot{x}}{|\dot{\mathbf{x}}|}F_L\end{aligned}$$

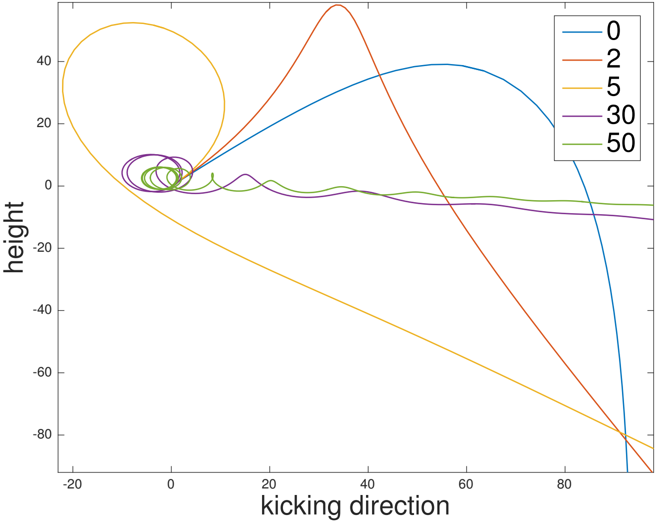
- The lift force F_L is a consequence of top or bottom spin applied to the ball on impact.
- We can analyse the effects of gravity, drag and spin on the path of the ball.

Drag and spin effects on vertical movement

InFoMM
Industrially Focused Mathematical Modelling

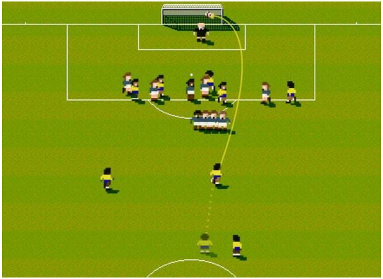


Varying effects of spinning



Birds eye view

- Looking from above, the Magnus force corresponds to a *lateral force*, which is a consequence of side-spin.



- We hence obtain the following model:

$$\ddot{x} = -\frac{1}{m}\frac{\dot{x}}{|\dot{\mathbf{x}}|}F_D - \frac{1}{m}\frac{\dot{y}}{|\dot{\mathbf{x}}|}F_L$$

$$\ddot{y} = -\frac{1}{m}\frac{\dot{y}}{|\dot{\mathbf{x}}|}F_D + \frac{1}{m}\frac{\dot{x}}{|\dot{\mathbf{x}}|}F_L$$

Reformulate..

- Write $K_D v^2 = \frac{F_D}{m}$ and $K_L v^2 = \frac{F_L}{m}$ to obtain

$$\ddot{x} = -K_D \dot{x}\sqrt{\dot{x}^2 + \dot{y}^2} - K_L \dot{y}\sqrt{\dot{x}^2 + \dot{y}^2}$$
$$\ddot{y} = -K_D \dot{y}\sqrt{\dot{x}^2 + \dot{y}^2} + K_L \dot{x}\sqrt{\dot{x}^2 + \dot{y}^2}$$
$$x(0) = 0 = y(0) \quad x'(0) = 0 \quad y'(0) = v$$

- We rescale:

$$t = \tau \hat{t} \qquad x = L_1 \hat{x} \qquad y = L_2 \hat{y}$$

and assume that for y only the drag is of relevance. (Distance towards goal much bigger than sideways motion.) Then we obtain

$$\ddot{\hat{x}} = -\dot{\hat{y}} \sqrt{1 + \frac{L_1^2}{L_2^2} \frac{\dot{\hat{x}}^2}{\dot{\hat{y}}^2}} \left(L_2 K_D \dot{\hat{x}} + \frac{L_2^2 K_L}{L_1} \dot{\hat{y}} \right)$$
$$\ddot{\hat{y}} = -\dot{\hat{y}}^2 L_2 K_D$$

The equation for $\hat{y}$ can be solved analytically and can be redimensionalized:

$$y(t) = \frac{1}{K_D} \log(1 + tvK_D)$$

Now considering the equation for $\hat{x}$, we use the following scalings:

$$L_1 = K_L L_2^2 \qquad \epsilon = K_D L_2$$

and write $a = \frac{K_L}{K_D}$. That gives

$$\begin{aligned} \ddot{\hat{x}} &= -\dot{\hat{y}}\sqrt{1 + a^2\epsilon^2\frac{\dot{\hat{x}}^2}{\dot{\hat{y}}^2}}\left(\epsilon\dot{\hat{x}} + \dot{\hat{y}}\right) \\ &\approx \dot{\hat{y}}\left(1 + \frac{1}{2}a^2\epsilon^2\frac{\dot{\hat{x}}^2}{\dot{\hat{y}}^2}\right)\left(\epsilon\dot{\hat{x}} + \dot{\hat{y}}\right) \end{aligned}$$

Now write $\hat{x} = \hat{x}^0 + \epsilon\hat{x}^1 + \ldots$ and obtain

$$\begin{aligned} \ddot{\hat{x}}^0 &= -\left(\dot{\hat{y}}\right)^2 \\ \ddot{\hat{x}}^1 &= -\dot{\hat{y}}\dot{\hat{x}}^0 \end{aligned}$$

(both with homogenous boundary condition at zero).

- These can be solved and we obtain our solutions:

$$\hat{x}^0 = \frac{K_L \log\left(tvK_D + 1\right)}{K_D^2} - \frac{tvK_L}{K_D}$$

$$\epsilon\hat{x}^1 = \frac{vK_L\left(t\log\left(tvK_D + 1\right) + \frac{2\log(tvK_D+1)}{vK_D} - t\right)}{K_D} - \frac{tvK_L}{K_D}$$

Quality of the Asymptotics

- Lets consider a kick at 30m/s with 3 revolutions per second.
- We compare the analytical solution to a numerical solution.

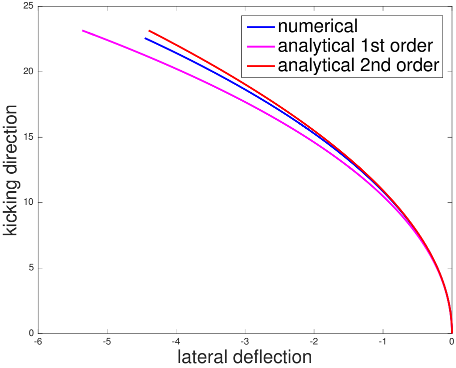
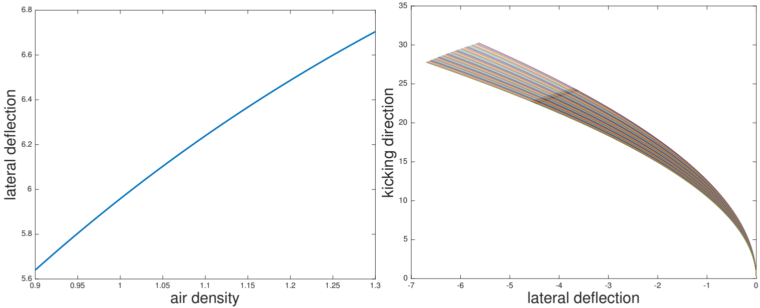


Figure: Numerical vs First-Order vs Second-Order approximation

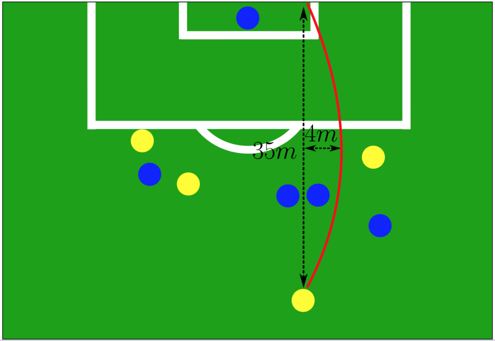
Location Changes

- Changes in air density can change the swerve of a typical free kick by up to a meter!



Back to Roberto Carlos...

- From video analysis we can infer that Robert Carlos freekick had a speed of approximately 30m/s and 10 revolutions per second.
- Our model gives a deviation from a straig shot of approximately 4m.
- This is coherent with measurements that sports journals have taken.



Drag crisis

- Low Reynolds number (i.e. low velocities): Laminar boundary layer separates from the ball earlier $\rightarrow$ forms wide, low-pressure wake $\rightarrow$ slows down ball.
- Higher Reynolds number (higher velocities): Boundary layer becomes turbulent $\rightarrow$ separates later than in laminar case $\rightarrow$ forms a narrower wake and a lower drag coefficient.
- A fast-moving ball may not slow down as quickly as a goalkeeper might expect.

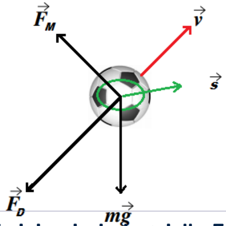
Other reasons for irratic behaviour

- Reverse Magnus effect:
  - The flight of a ball with a low angular velocity (relative to becomes primarily dependent on the air pressure which is constantly changing. The points where the flow transitions from laminar to turbulent are different on either side of the ball, so the Magnus effect can reverse, leading to a 'knuckle ball' effect.
  - The smoothness of the ball changes the position of these transition points: A ball with a rougher surface or stitching is more likely to follow the 'standard' positive Magnus effect.

$3D$ Model

$$\underline{x}(0) = (0,0,0)$$
$$\underline{\dot{x}}(0) = (5,0,5)$$
$$\underline{s} = (0,1,0)$$

$$\begin{pmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{pmatrix} = -g\begin{pmatrix}0\\0\\1\end{pmatrix} - \frac{F_d}{m\sqrt{(x^2+y^2+z^2)}}\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} + \frac{F_m}{m}\begin{pmatrix}\underline{s}\end{pmatrix}\times\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix}$$



3D plots

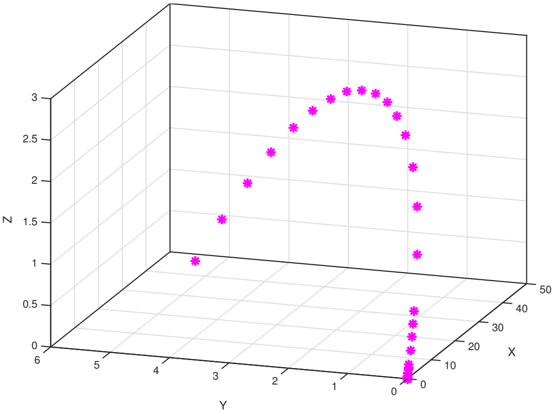


Figure: Spiral projection

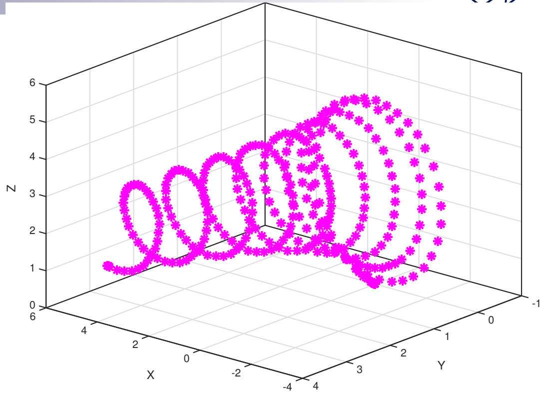


Figure: Drag crisis

Drag Crisis

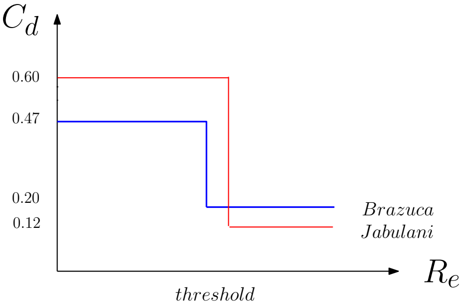


Figure: Drag crisis

Drag Crisis

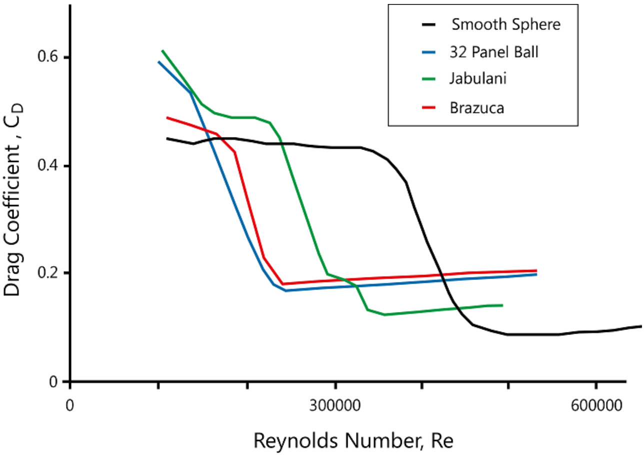


Figure: Drag crisis

2D projections

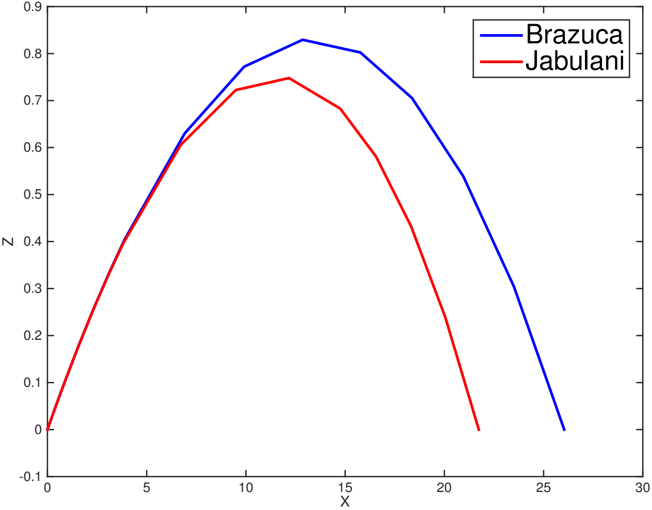


Figure: Comparison of drag crisis effect

Conclusions — What have we done?

- 2D Models for side view and birds eye view.
- Asymptotic solution for the birds eye model case when the deviation from a straight shot is small.
- Numerical solution for both models including the drag crisis.
- 3D Model including drag crisis.
- Numerical solution for 3D model.

Objectives

- Why does the ball swerve?
  - Magnus effect.
- Can we model the flight of a football?
  - Yes. In 2D (numerical & analytical) and 3D (numerical).
- Does the location matter?
  - Yes. The density of the air can change the amount of swerve drastically.
- Does the choice of ball matter?
  - Yes. Smoother balls have the drag crisis at higher velocities.
- Can we explain oscillations in the flight?
  - Only speculate...

Next steps

- Higher order asymtotics.
- 3D analytical solution.
- Better drag crisis model.
- Full CFD.
- Model any other famous freekicks?